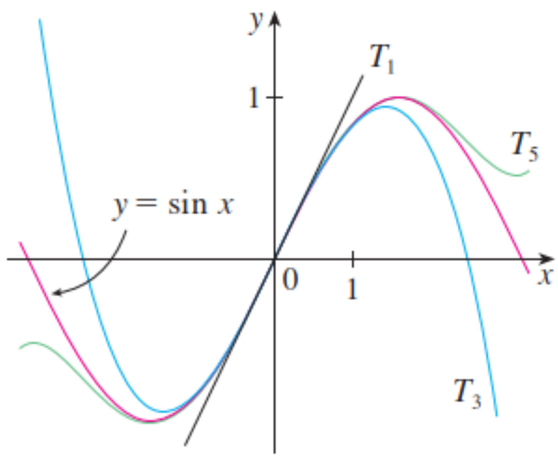
MATH 147 Review: Taylor Series and Radius of Convergence

Facts to Know

The Taylor Series of a function $f(x)$ centered at a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{holds for } x \in (a-R, a+R)$$

$$= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots$$



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

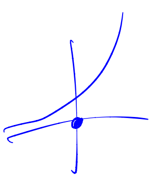
$T_1(x) = x$

$T_3(x) = x - \frac{x^3}{3!}$

$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

Examples

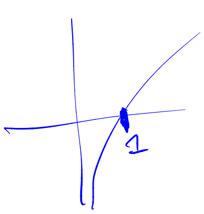
1. Find the Taylor Series of the function $f(x) = e^x$ centered at 0.



$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

2. Find the Taylor Series of the function $f(x) = \ln(x)$ centered at 1.



$$\ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{1}{1!}(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{+1}{3!}(x-1)^3 + \frac{-1}{4!}(x-1)^4$$

$f^{(0)} = \ln(x) \Rightarrow 0$

$f^{(1)} = \frac{1}{x} = x^{-1} \Rightarrow 1$

$f^{(2)} = \frac{-1}{x^2} \Rightarrow -1$

$f^{(3)} = \frac{+1}{x^3} \Rightarrow +1$

$f^{(4)} = \frac{-1}{x^4} \Rightarrow -1$

$$\ln(x+1) = \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{+1}{3!}x^3 + \frac{-1}{4!}x^4 + \dots$$

3. Find the Taylor Series of the function $f(x) = \frac{1}{1-x}$ centered at 0.

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad R = 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n$$

4. Find the Taylor Series of the function $f(x) = \frac{1}{(1-x)^2}$ centered at 0.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$